Comments on Higher Derivative Terms for the Tachyon Action

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Abstract

We consider the open string tachyon action in a world–sheet sigma model approach. We present explicit calculations up to order 8 in derivatives for the bosonic string, and mimic these to order 6 for the superstring, including terms with multiple derivatives acting on the tachyon field. We reproduce lower derivative terms obtained elsewhere, and speculate on the role of the world–sheet contact terms in regularizing the action for the superstring tachyon.

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1 Introduction

The existence of a tachyon in a field theory is generally understood to be indicative of an instability in the theory; that the theory is being expanded around an incorrect vacuum. There has therefore been a great deal of interest in investigating the dynamics associated with the tachyonic degrees of freedom that appear in the spectrum of both bosonic and superstring theories. The condensation of the open string tachyon into a stable vacuum remains an interesting open problem addressed in off–shell string theory.

Some of the original work on this subject [1, 2, 3, 4] was concerned with obtaining an action for the tachyon field by integrating out stringy degrees of freedom to obtain a (string) partition function in which the couplings were related to the target space fields. This partition function was to be interpreted as a space time action for the fields thus described, and the efforts were in parallel with the derivation of the Born–Infeld action from strings interacting with a background gauge field [5]. The tachyon profiles originally considered were manifestly off–shell. Their particular form made the world–sheet theory solvable, and exact expressions for the action were obtained in this regime [2, 6, 7, 8]. This line of inquiry was generalized to the superstring [9] and it has been understood that even such simple profiles were useful toy models in the study of D–brane decay [10, 11, 12]. A more recent approach to this has been to study the branes described by the 'rolling tachyon' solution [13, 14]. This solution is an exactly marginal perturbation on the string world–sheet boundary and describes the time–dependent decay of a brane. It has been shown that, when coupled to a gauge field, the decay products of this process is a radiation of closed strings carrying the polarization of the coupled magnetic field [15, 16].

Invoking other considerations it has also been argued that what is commonly called the 'tachyon DBI action',

$$S = \int V(T)\sqrt{\det\left(g_{\mu\nu} + \partial_\mu T\partial_\nu T + \partial_\mu \Phi^i \partial_\nu \Phi_i\right)} \tag{1}$$

reproduces the dynamics for the D–brane system with transverse scalars Φ_i. This action has been the subject of numerous studies, for example [17, 14, 18, 19, 20], and describes an aspect of the tachyon interaction with other degrees of freedom, even in the off–shell case. The tachyon DBI action is also the Lorentz covariant generalization of an exact solution of the tachyon action from the rolling tachyon [21]. In an attempt to better understand the off–shell dynamics of the string, it has been proposed to superimpose a spatial dependence on the rolling tachyon solution [22] and examine

the corresponding action. For certain (periodic) spatial dependences exact solutions have been found [23, 24]. The more frequent practice follows the calculation of derivative corrections to the Born–Infeld action [25] using an expansion around a marginal boundary interaction. The action is understood to be accurate to a certain order in spatial derivatives.

A different approach which is much less reliant on world–sheet properties is to extract on–shell matrix elements for different scattering processes [26]. Once these are calculated the appropriate terms in the action can then be reconstructed. One difficulty with this approach is that the starting point is an on–shell calculation, and so is subject to ambiguities when one tries to extend to off–shell processes (for a recent discussion, see [27]).

An interesting paper [22] raised a number of points with respect to these lines of investigation: The argument which established the tachyon DBI action as an exact solution for the rolling tachyon [21] does not trivially extend to spatial variations, and so motivates a search for the Lorentz invariant generalization. The second was to ask the question about the meaning of the tachyon action, obtained by integrating out the stringy modes: In the case of a massless particle the higher stringy modes are separated by a mass gap, however for the tachyon the (negative) mass squared is of the same order as the infinite tower which introduces subtleties to the interpretation of the effective action. The third point was that the tachyon DBI action depends only on first derivatives of the tachyon field. It was argued that a consistent expansion should also contain higher order derivatives on the tachyon field, as obtained through integration by parts.

We are motivated by the argument that the tachyon action should be Lorentz invariant and contain all possible combinations of derivatives, and in this paper we attempt to address these points following the world–sheet techniques of [28]. We find that the technique has an obvious generalization to problems like the expansion around the rolling tachyon solution. It also offers some lessons about the derivative structure that is 'natural' from the world–sheet point of view: that for off–shell processes there is a non–local field redefinition which depends on the normal ordering prescription for the sigma model. Related to this, the action does not have any terms which depend on the derivatives $\partial_\mu\partial^\mu$ acting on a single field. We are also motivated to understand the contact term in the boundary theory on the superstring theory. This term is required for unbroken world–sheet supersymmetry [9], but this term is omitted from the analysis in [21]. The omission was explained in [22] as being justified in the momentum region under consideration. We speculate that this contact term is necessary to appropriately regularize the tachyon action in particular momentum regions.

This paper is organized as follows, in section 2 we review the derivation of the action for the boundary tachyon coupled to a bosonic string. This will have an obvious generalization to the case of superstring theory, and will illustrate the mechanics of our calculation. In section 3 we illustrate the calculation for the superstring using a superfield formalism, and find an action for the tachyon up to 6 derivatives. In section 4 we summarize and discuss our results. A number of the technical details, including the regularization prescription, are relegated to appendix A.

2 Effective action for bosonic string tachyon

Prior to performing the calculation for the superstring, we present a derivation of the derivative expansion for the bosonic open string tachyon action. We do this to illustrate the technique used without the additional complication of fermion fields on the string world–sheet, and also because the origin of some features that are reproduced in the superstring are easier to trace in this simplified setting.

The method used here is inspired by the worldsheet approach of [5]. Specifically we integrate out the world–sheet degrees of freedom and obtain a partition functional of the tachyon field. We expand to a fixed order in derivatives of the tachyon field and interpret the resulting expression as the first terms of a covariant expression for the tachyon action.

We introduce the tachyon field as a boundary interaction of the unit disk with the action

$$S = \frac{1}{2\alpha'}\int\frac{d^2z}{2\pi}\partial X^{\mu}\bar{\partial}X_{\mu}+\oint\frac{d\phi}{2\pi}T(X(\phi)). \tag{2}$$

In (2) ϕ is the coordinate on the boundary of the world–sheet. We then divide the field X into a classical and oscillatory part as $X\to x+\tilde{X}$, and write the mode expansion as

$$T(X)=\int dkT(k)e^{ikx}e^{ik\tilde{X}}. \tag{3}$$

We now follow [2, 29] and calculate a partition functional that depends on the tachyon by integrating out the oscillatory modes $\tilde{X}$ using the action (2), explicitly

$$Z\left(T(x)\right) = \int d\tilde{X}e^{-S\left(T(x+\tilde{X})\right)}. \tag{4}$$

Once we have obtained the partition functional for the bosonic string, we may calculate the space–time action as

$$S\left(T(x)\right) \;\; = \;\; \left(\beta_T \partial_T + 1\right) Z\left(T(x)\right). \tag{5}$$

We expand Z as a power series in T and find that at order T^n we have

$$\frac{(-1)^n}{n!} \oint \prod_{i=1}^{n} \frac{d\phi_i}{2\pi} \langle \prod_{i=1}^{n} T\left(x + \tilde{X}(\phi_i)\right) \rangle. \tag{6}$$

This can be evaluated as

$$\frac{(-1)^n}{n!} \oint \prod_{i=1}^{n} \frac{d\phi_i}{2\pi} \left[\prod_{i=1}^{n} \int dk_i e^{ik\cdot x} T(k_i) \right] \prod_{i=1}^{n} e^{-\frac{k_i^2}{2} B(0)} \prod_{i<j=1}^{n} e^{-k_i \cdot k_j B(\phi_i - \phi_j)}, \tag{7}$$

where $B(\phi - \phi')$ is the propagator for the oscillatory modes, given by

$$\begin{aligned} B(\phi - \phi') &\equiv \langle \tilde{X}^\mu(\phi) \tilde{X}^\nu(\phi') \rangle & (8) \\ &= -2\alpha' g^{\mu\nu} \ln \left| 2 \sin \frac{\phi - \phi'}{2} \right| \\ &= 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\cos m(\phi - \phi')}{m}. & (9) \end{aligned}$$

Our convention for the metric is that the signature of $g^{\mu\nu}$ is $(-, + \ldots +)$. The expression (7) is nothing but an off–shell bosonic string tachyon scattering amplitude for general values of k and $T(k)$. Due to the off–shell nature of this calculation we have retained a term associated with self contractions of the tachyon vertex operators. For the same reason we have not used conformal invariance to fix the location of any of the tachyon insertions. We absorb the self contractions by a non–local field redefinition

$$\tilde{T}(x) \;\; = \;\; e^{c\partial_\mu \partial^\mu} T(x), \tag{10}$$

where c is a constant associated with the regularized version of $B(0)$ and defined by $c = \frac{B(0)_{\rm reg}}{2}$. This term appears in some other analyses, such as [30, 31], and on–shell it is simply a rescaling, but off–shell it amounts to redefining the tachyon field to include the anomalous dimension.

We can obtain an exact result for the integral (7) when we consider a tachyon background with a single momentum mode and profile $T_p(X) = T_0 e^{ip\cdot X}$. In this case the integral is trivially recast as Dyson's integral, a

special case of Selberg's generalization of the β–function [32]. Then we obtain for n insertions of T

$$
\begin{aligned}
\frac{(-1)^n}{n!}\left[\tilde{T}_p(x)\right]^n \oint & \prod_{i=1}^{n}\frac{d\phi_i}{2\pi}\prod_{i<j=1}^{n} e^{-p^2 B(\phi_i-\phi_j)} \\
&= \frac{(-1)^n}{n!}\left[\tilde{T}_p(x)\right]^n \oint \prod_{i=1}^{n}\frac{d\phi_i}{2\pi}\prod_{i<j=1}^{n}\left|e^{i\phi_i}-e^{i\phi_j}\right|^{2\alpha' p^2} \\
&= \frac{(-1)^n}{n!}\left[\tilde{T}_p(x)\right]^n \frac{\Gamma(1+n\alpha' p^2)}{\Gamma(1+\alpha' p^2)^n}. \qquad (11)
\end{aligned}
$$

This expression interpolates between two known cases, the expansion around the constant tachyon which gives rise to a potential e^{-T_p} [2], and the expansion around the rolling tachyon which gives a potential $\frac{1}{1+T_p}$ [21]. The correlation function (11) can also be expanded in perturbation around this plane wave solution using $p^2 \to p^2+\delta p^2$. The variation about the constant tachyon clearly vanishes at first order in δp^2. About the $p^2 = 1$ rolling tachyon case it produces $\left[\tilde{T}_p(x)\right]^n n\left(\psi(1+n)-\psi(2)\right)\delta p^2$, and re-casting the momentum as derivatives, and summing over n reproduces the spatial derivative term of [22]. The case $n = 2$ of (11) can be immediately extended to the well–known expression for the two–point function for the bosonic tachyon [30, 8],

$$
\frac{\tilde{T}^2}{2!}\frac{\Gamma[1+2\alpha' k_1\cdot k_2]}{\Gamma^2[1+\alpha' k_1\cdot k_2]} \qquad (12)
$$

and in [30] the three–point function for the bosonic tachyon is derived to be

$$
\frac{\tilde{T}^3}{3!}\Gamma(1+k_{12}+k_{13}+k_{23})\prod_{i<j=1}^{3}\frac{\Gamma(1+2k_{ij})}{\Gamma(1+k_{ij})\Gamma(1+k_{12}+k_{13}+k_{23}-k_{ij})} \qquad (13)
$$

where $k_{ij} = \alpha' k_i \cdot k_j$. (13) is consistent with (11) in the case of a single momentum mode.

We wish to understand the application of these exact expressions we consider the region around the constant tachyon field and our strategy is to expand the expression (7) in powers of k, do the integration over the insertion points for the field T on the world–sheet boundary, and then to Fourier transform back to position space. This identifies the expansion in powers of k with an expansion in powers of derivatives, and is similar to the strategy used in [25] to obtain the Born–Infeld action and derivative corrections from a world–sheet approach. We note that do not expand around a

Prefactor	Derivative Form	Associated Sum
-1	T	
$\frac{1}{8}(2\alpha')^2$	$\partial_{\mu\nu}T\partial^{\mu\nu}T$	$\frac{1}{m^2}$
$-\frac{1}{16}(2\alpha')^3$	$\partial_{\mu\nu\alpha}T\partial^{\mu\nu\alpha}T$	$\frac{1}{mn(m+n)}$
$-\frac{1}{3\cdot 4}(2\alpha')^3$	$\partial_\mu{}^\nu T\partial_\nu{}^\alpha T\partial_\alpha{}^\mu T$	$\frac{1}{m^3}$
$\frac{1}{3!16}(2\alpha')^4$	$\partial_{\mu\nu\alpha\beta}T\partial^{\mu\nu\alpha\beta}T$	$\frac{4}{mnn'(m+n+n')}+\frac{3\delta_{m+m',n+n'}}{mm'nn'}$
$-\frac{1}{32}(2\alpha')^4$	$\partial_{\mu\nu}T\partial^\mu{}_{\alpha\beta}T\partial^{\nu\alpha\beta}T$	$\frac{2}{n^2m(m+n)}+\frac{1}{mn(m+n)^2}$
$-\frac{1}{32}(2\alpha')^4$	$\partial_{\mu\nu}T\partial^{\mu\nu\alpha\beta}T\partial_{\alpha\beta}T$	$\frac{1}{m^2n^2}$
$\frac{1}{32}(2\alpha')^4$	$\partial_\mu{}^\nu T\partial_\nu{}^\alpha T\partial_\alpha{}^\beta T\partial_\beta{}^\mu T$	$\frac{1}{n^4}$
$(-1)^k\frac{1}{k2^{k-1}}(2\alpha')^k$	$\mathrm{Tr}(\partial_\mu{}^\nu T)^k$	$\frac{1}{m^k}$

Table 1: The terms which appear in the tachyon action for bosonic string theory around the constant tachyon, with their prefactors and the associated sums. In all sums implicitly over the positive integers, and we use the abbreviation $\partial_\mu\partial_\nu \equiv \partial_{\mu\nu}$ with generalizations to higher orders in derivatives.

conformal solution, because while $F_{\mu\nu} =$ constant is on–shell, $T =$ constant is not. Using standard field theory techniques and the propagator (9), it is straightforward to perform the necessary worldsheet integrals. In table 1 we give the prefactors, sums, and derivative structure that appear.

In the appendix we give some more details of this calculation. To order 8 in derivatives we obtain a partition functional given by

$$
\begin{aligned}
Z\left(T(x)\right) \quad = \quad \exp\Big[& -\tilde{T}(x) + \frac{\zeta(2)}{2}\alpha'^2\partial_{\mu\nu}\tilde{T}(x)\partial^{\mu\nu}\tilde{T}(x) \\
& + \zeta(3)\alpha'^3\partial_{\mu\nu\gamma}\tilde{T}(x)\partial^{\mu\nu\gamma}\tilde{T}(x) \\
& + \frac{1}{4}\left(\zeta^2(2) + 7\zeta(4)\right)\alpha'^4\partial_{\mu\nu\gamma\delta}\tilde{T}(x)\partial^{\mu\nu\gamma\delta}\tilde{T}(x) \\
& - \frac{\zeta(3)}{3}\alpha'^3\partial_\mu{}^\nu\tilde{T}(x)\partial_\nu{}^\gamma\tilde{T}(x)\partial_\gamma{}^\mu\tilde{T}(x) \\
& - \frac{1}{2}\zeta^2(2)\alpha'^4\partial_{\mu\nu\gamma\delta}\tilde{T}(x)\partial^{\mu\nu}\tilde{T}(x)\partial^{\gamma\delta}\tilde{T}(x) \\
& + \left(\zeta^2(2) + \frac{1}{2}\zeta(4)\right)\alpha'^4\partial_{\mu\nu\gamma}\tilde{T}(x)\partial^{\mu\nu}{}_\delta\tilde{T}(x)\partial^{\gamma\delta}\tilde{T}(x) \\
& + \frac{1}{4}\zeta(4)\alpha'^4\partial_\mu{}^\nu\tilde{T}(x)\partial_\nu{}^\gamma\tilde{T}(x)\partial_\gamma{}^\delta\tilde{T}(x)\partial_\delta{}^\mu\tilde{T}(x) + \mathrm{O}(\partial^{10})\Big]\,.
\end{aligned}
\tag{14}
$$

Here and throughout this paper whenever multiple derivatives act on the

same field we abbreviate $\partial_\mu \partial_\nu \equiv \partial_{\mu\nu}$. Inspecting the expression (14) we note that the terms $\frac{\zeta(n)}{n}(\partial_\mu{}^\nu \tilde{T})^n$ that are familiar from the discussion of the quadratic tachyon profile [2, 6, 12] appear. In addition to this, the field redefinition (10) that is imposed by the world–sheet sigma model calculation gives a prescription for resolving some of the ambiguities in defining an off–shell tachyon action discussed in [22]. The substitution of $\tilde{T}$ for T throughout reproduces that world–sheet cut–off dependent coefficient. The Taylor expansion of (10) in powers of the cutoff–dependent coefficient exactly matches the first two terms of the renormalized field of [28]. Equivalently, the partition functional (14) can be expressed in terms of the original field $T(x)$ as

$$
\begin{aligned}
Z\left(T(x)\right) \;=\; e^{-T}\Bigg[& 1 + c\partial_\mu{}^\mu T - \frac{c^2}{2!}\partial_{\mu\nu}{}^{\mu\nu} T + \frac{c^3}{3!}\partial_{\mu\nu\alpha}{}^{\mu\nu\alpha} T \\
& + \frac{c^2}{2!}\partial_\mu{}^\mu T \partial_\nu{}^\nu T - \frac{c^3}{2!}\partial_\mu{}^\mu T \partial_{\nu\alpha}{}^{\nu\alpha} T + \frac{c^3}{3!}\partial_\mu{}^\mu T \partial_\nu{}^\nu T \partial_\alpha{}^\alpha T \\
& + \frac{\zeta(2)}{2}\alpha'^2 \partial_{\mu\nu} T \partial^{\mu\nu} T - \zeta(2) c \alpha'^2 \partial_{\mu\nu\alpha}{}^{\alpha} \partial^{\mu\nu} T \\
& + \zeta(3)\alpha'^3 \partial_{\mu\nu\gamma} T \partial^{\mu\nu\gamma} T - \frac{\zeta(3)}{3}\alpha'^3 \partial_\mu{}^\nu T \partial_\nu{}^\gamma T \partial_\gamma{}^\mu T + \mathrm{O}(\partial^8)\Bigg] .
\end{aligned}
\tag{15}
$$

Using this expression, we could follow [29] and fix the value of c to give a conventionally normalized kinetic term. The expression in terms of $\tilde{T}$ (14) does not contain any two–derivative terms, and appears more natural to us. This situation is somewhat different than the superstring case, which is discussed in section 3.

The term quadratic in the tachyon field (12) can be Taylor expanded in $k_1 \cdot k_2$, giving

$$
\frac{\tilde{T}^2}{2}\left(1 + \zeta(2)(\alpha' k_1 \cdot k_2)^2 - 2\zeta(3)(\alpha' k_1 \cdot k_2)^3 + \frac{19\pi^2}{360}(\alpha' k_1 \cdot k_2)^4 + O(k^{10})\right), \tag{16}
$$

which exactly matches the terms quadratic in $\tilde{T}$ in the expansion of (14). In the case of the three point function, fixing two of the k_{ij}s in (13) to be zero reduces it to (12) times $\tilde{T}$, the terms in its expansion in a single k_{ij} matches that of (14). Similarly, the expansion of (13) to linear order in two of the k_{ij} vanishes, while the term proportional to $k_{12}k_{13}k_{23}$ is $\frac{1}{3!}\psi''(1) = -\frac{1}{3}\zeta(3)$, in agreement with (14). The coefficient proportional to $k_{12}k_{13}k_{23}^2$ is $\frac{3}{2}\psi'''(1)$ from expansion of (13) which matches the $\frac{\pi^4}{30}$ from (14). We emphasize

that the analytic expressions obtained in (12) and (13) are valid for $k^2 > 0$, and that the perturbative expression is valid for $|k^2| \ll 1$. This fact that the derivative expansion is consistent with the results in two regimes, leads us to conjecture that both correctly capture the two–point and three–point dynamics of open string tachyons. We shall see that the situation for the two–point function is similar, but not exactly the same in the case of the superstring.

3 Effective action for superstring tachyon

We use the intuition from the calculation in the bosonic sector to proceed in the case of the superstring, and outline the calculation below. Many of the intermediate steps are detailed in the appendix. We start with the conventions of [9] for the action, using their superfield method to obtain the higher derivative terms for the tachyon partition function. We define superfields and a derivative by

$$\begin{aligned} \Gamma(\phi,\theta) &= \eta(\phi) + \theta F(\theta), \\ \mathbf{X}^\mu(\phi,\theta) &= X^\mu(\phi) + \theta\psi^\mu(\phi), \\ D &= \partial_\theta + \theta\partial_\phi. \end{aligned} \tag{17}$$

The boundary action coupled to a tachyon field is

$$S_{boundary} = \oint \frac{d\phi}{2\pi} d\theta [\Gamma D \Gamma + T(\mathbf{X})\Gamma]. \tag{18}$$

The partition function is then given by

$$Z = \int [d\mathbf{X}][d\Gamma] e^{-S_{bulk} - S_{boundary}} = \int [d\mathbf{X}][d\Gamma] e^{-S_0 - S_1}, \tag{19}$$

with S_{bulk} the standard NSR action in the bulk, and with the boundary superfield kinetic term included in S_0 giving

$$\begin{aligned} S_0 &= S_{bulk} + \oint \frac{d\phi}{2\pi} d\theta \Gamma D \Gamma, \\ S_1 &= \oint \frac{d\phi}{2\pi} T(\mathbf{X})\Gamma. \end{aligned} \tag{20}$$

In the superstring case, the tachyon action is equal to the partition function [9]. As in the bosonic case we expand into a constant mode and fluctuations,

with $\mathbf{X} = x + \tilde{\mathbf{X}}$, where x denotes a constant mode and $\tilde{X}$ encodes non-zero modes, and we do a Fourier transformation for the tachyon field

$$\begin{aligned} T(\mathbf{X}) &= \int dk f(k) e^{ik\cdot\mathbf{X}} = \int dk f(k) e^{ik\cdot x} e^{ik\cdot\tilde{\mathbf{X}}}, \\ S_1 &= \int dk f(k) e^{ik\cdot x} \oint \frac{d\phi}{2\pi} d\theta e^{ik\cdot\tilde{\mathbf{X}}} \Gamma. \end{aligned} \tag{21}$$

We can expand the partition function as

$$\begin{aligned} Z = \int dx \int [d\tilde{\mathbf{X}}][d\Gamma] e^{-S_0 - S_1} &= \int dx \int [d\tilde{\mathbf{X}}][d\Gamma] e^{-S_0} \sum_{n=0}^{\infty} \frac{1}{n!} (-S_1)^n \\ &= \int dx \sum_{n=0}^{\infty} \frac{1}{n!} \langle (-S_1)^n \rangle. \end{aligned} \tag{22}$$

We note that since S_1 is proportional to Γ, $\langle (-S_1)^n \rangle$ is non-vanishing only for even n, which allows us to write the partition function as

$$Z = Z_0 + Z_2 + Z_4 + Z_6 + \dots, \tag{23}$$

where the subscripts indicate the number of tachyon fields in each term. The nth order term is given by

$$\begin{aligned} Z_n &= \frac{1}{n!} \int dx \langle (-S_1)^n \rangle \\ &= \frac{1}{n!} \int dx \int \prod_{i=1}^{n} dk_i f(k_i) e^{i(k_1 + \cdots k_n)\cdot x} \oint \prod_{i=1}^{n} \frac{d\phi_i}{2\pi} \\ &\quad \int d\theta_n \cdots d\theta_1 \langle e^{ik\cdot\tilde{\mathbf{X}}(1)} \cdots e^{ik\cdot\tilde{\mathbf{X}}(n)} \rangle \langle \Gamma(1) \cdots \Gamma(n) \rangle. \end{aligned} \tag{24}$$

We have abbreviated the ϕ_i, θ_i dependence by making their subscript i the argument. The superfield propagators are

$$\begin{aligned} \langle \tilde{\mathbf{X}}(\phi_1, \theta_1) \tilde{\mathbf{X}}(\phi_2, \theta_2) \rangle &= \langle \tilde{X}(\phi_1) \tilde{X}(\phi_2) \rangle - \theta_1 \theta_2 \langle \psi(\phi_1) \psi(\phi_2) \rangle, \\ \langle \Gamma(\phi_1, \theta_1) \Gamma(\phi_2, \theta_2) \rangle &= \langle \eta(\phi_1) \eta(\phi_2) \rangle + \theta_1 \theta_2 \langle F(\phi_1) F(\phi_2) \rangle, \end{aligned} \tag{25}$$

with boundary-to-boundary propagators for the component fields given by

$$
\begin{aligned}
\langle \tilde{X}(\phi_1)\tilde{X}(\phi_2)\rangle &= \alpha' \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{1}{|m|} e^{im(\phi_1-\phi_2)}, \\
\langle \psi(\phi_1)\psi(\phi_2)\rangle &= \alpha' i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{r}{|r|} e^{ir(\phi_1-\phi_2)}, \\
\langle F(\phi_1)F(\phi_2)\rangle &= \frac{1}{2}\sum_{m\in\mathbb{Z}} e^{im(\phi_1-\phi_2)}, \\
\langle \eta(\phi_1)\eta(\phi_2)\rangle &= \frac{1}{2} i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r} e^{ir(\phi_1-\phi_2)}. \qquad (26)
\end{aligned}
$$

As in the bosonic case, the propagator at zero separation for the field $\tilde{X}$ is divergent. The propagator of $F(\phi)$ is a δ-function $\pi\delta(\phi_1-\phi_2)$ for periodic functions and the η propagator is a step function. We can evaluate (24) and obtain the n–point function

$$
\langle e^{ik_1\cdot\tilde{\mathbf{X}}(1)} \ldots e^{ik_n\cdot\tilde{\mathbf{X}}(n)}\rangle = e^{-({k_1}^2+\ldots+{k_n}^2)c} \prod_{i<j=1}^{n} e^{-k_i\cdot k_j \langle \tilde{\mathbf{X}}(i)\tilde{\mathbf{X}}(j)\rangle} \qquad (27)
$$

$$
\begin{aligned}
\langle \Gamma(1)\ldots\Gamma(n)\rangle &= \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \langle \Gamma(\mathcal{P}(1))\Gamma(\mathcal{P}(2))\rangle \\
&\qquad \ldots \langle \Gamma(\mathcal{P}(n-1))\Gamma(\mathcal{P}(n))\rangle \qquad (28)
\end{aligned}
$$

The sum in (28) is over the $(n-1)!!$ pairwise distinct permutations of the indices $1\ldots n$. The c in (27) is the same as that in (10), and those terms will be used rescale T to $\tilde{T}$. Substituting (27) and (28) into (24) and integrating over the Grassman coordinates is equivalent to using the derived expression for the boundary interaction that was obtained by taking (19) and integrating out the boundary superfield. The action given by [9] is

$$
\begin{aligned}
S &= \frac{1}{2\alpha'}\int \frac{d^2z}{2\pi}\Big(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\Big) + \frac{1}{4}\oint \frac{d\phi}{2\pi} T^2\left(X(\phi)\right) \\
&\quad + \frac{1}{4}\oint \frac{d\phi d\phi'}{4\pi}\epsilon(\phi-\phi')\Big(\psi^\mu(\phi)\partial_\mu T(X(\phi))\Big)\Big(\psi^\mu(\phi')\partial_\mu T(X(\phi'))\Big), \qquad (29)
\end{aligned}
$$

where the function $\epsilon(x)$ is the sign function, so that $\epsilon(x)=1$ for $x>0$ and $\epsilon(x)=-1$ for $x<0$. As we consider the NS-sector, this relates ψ

and $\tilde{\psi}$ on the boundary. This interactions features both a non–local fermion interaction and a contact term proportional to T^2.

There are two well known exact expressions for the superstring tachyon for tachyons with a single momentum mode like those discussed in the context of the bosonic string. The first is the expansion around the constant tachyon, which gives rise to a potential $e^{-\frac{\tilde{T}^2}{4}}$ [9], and the second is the expansion around the rolling tachyon with $k^2 = \frac{1}{2}$ which has a potential of $\frac{1}{1+\frac{1}{2}\tilde{T}^2}$ [21]. In fact, for any momentum, the term bilinear in the tachyon couplings can be written as

$$\begin{aligned}\langle\frac{1}{2!}\oint T(\phi_1)T(\phi_2)\rangle &= \frac{1}{2}\int dk_1 dk_2 f(k_1)f(k_2)e^{i(k_1+k_2)\cdot x}e^{-\frac{1}{2}(k_1^2+k_2^2)\delta} \\ &\quad\times\oint\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}d\theta_2 d\theta_1\langle\Gamma(1)\Gamma(2)\rangle e^{-k_1\cdot k_2\langle\tilde{\mathbf{X}}(1)\tilde{\mathbf{X}}(2)\rangle}\end{aligned} \tag{30}$$

The momentum dependence is encoded in the integral

$$\begin{aligned}&\oint\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}d\theta_2 d\theta_1\Big(\langle\eta(1)\eta(2)\rangle+\theta_1\theta_2\langle F(1)F(2)\rangle\Big) \\ &\qquad\times e^{-\alpha' k_1\cdot k_2\left(\langle\tilde{X}(1)\tilde{X}(2)\rangle-\theta_1\theta_2\langle\psi(1)\psi(2)\rangle\right)} \quad = \\ &\oint\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}\left(\pi\delta(\phi_1-\phi_2)+2\alpha' k_1\cdot k_2\frac{\epsilon(\phi_1-\phi_2)}{2\sin\frac{\phi_1-\phi_2}{2}}\right) \quad\times\quad \left|2\sin\frac{\phi_1-\phi_2}{2}\right|^{2\alpha' k_1\cdot k_2}\end{aligned} \tag{31}$$

For $k_1\cdot k_2 > 0$, the contact term in (31) vanishes, and we are left with

$$\langle\frac{1}{2!}\oint T(\phi_1)T(\phi_2)\rangle = \frac{\tilde{T}^2}{2!}\pi\frac{\Gamma(1+2\alpha' k_1\cdot k_2)}{\Gamma^2(\frac{1}{2}+\alpha' k_1\cdot k_2)} \tag{32}$$

while for $k_1\cdot k_2 < 0$ both integrals diverge and must be defined by regularization, as we discuss below.

In analogy with the calculation outlined for the bosonic string we can calculate an expansion in powers of momentum around the background of a constant tachyon. The expansion parameter is the momentum, which is presumed to be small, so we expand the exponential in k, which will be Fourier transformed into powers of derivatives as before. In table 2 we list the sums and derivative structures that appear in the Taylor expansion of (19) about $k = 0$.

From the explicit formulae for Z_n, we can extract an expression for the partition functional. After some manipulations, the expression for the

partition functional of $\tilde{T}$ is given as

$$
\begin{aligned}
Z\left(\tilde{T}(x)\right) \quad = \quad & \exp\left(-\frac{1}{4}\tilde{T}^2 + \alpha' \ln 2\partial^\mu\tilde{T}\partial_\mu\tilde{T} + \alpha'^2\left(\frac{1}{2}\zeta(2) - 2(\ln 2)^2\right)\partial^{\mu\nu}\tilde{T}\partial_{\mu\nu}\tilde{T}\right. \\
& +\alpha'^3\left(\frac{1}{2}\zeta(3) - 4\zeta(2)\ln 2 + \frac{8}{3}(\ln 2)^3\right)\partial^{\mu\nu\alpha}\tilde{T}\partial_{\mu\nu\alpha}\tilde{T} \\
& +\alpha'^2\frac{1}{2^5}\zeta(2)\partial^{\mu\nu}\tilde{T}^2\partial_{\mu\nu}\tilde{T}^2 - \alpha'^2\frac{3}{8}\zeta(2)\partial^\mu\tilde{T}\partial_\mu\tilde{T}\partial^\nu\tilde{T}\partial_\nu\tilde{T} \\
& +\alpha'^3\frac{1}{2^4}\zeta(3)\partial^{\mu\nu\alpha}\tilde{T}^2\partial_{\mu\nu\alpha}\tilde{T}^2 - \alpha'^3\frac{1}{2^4}\zeta(2)\ln 2\partial^\mu\tilde{T}\partial_{\mu\nu\alpha}\tilde{T}\partial^{\nu\alpha}\tilde{T}^2 \\
& +\alpha'^3\frac{1}{8}\left(7\zeta(3) - 4\ln 2\right)\partial^{\mu\nu}\tilde{T}\partial^\alpha_{\ \mu}\tilde{T}\partial_{\nu\alpha}\tilde{T}^2 \\
& +\alpha'^3\frac{7}{4}\zeta(3)\partial^\mu\tilde{T}\partial_{\mu\nu}\tilde{T}\partial^{\nu\alpha}\tilde{T}\partial_\alpha\tilde{T} - \alpha'^3\frac{1}{2^5 3}\zeta(3)\partial_{\mu\nu}\tilde{T}^2\partial^{\nu\alpha}\tilde{T}^2\partial_\alpha^{\ \mu}\tilde{T}^2 \\
& +\alpha'^3\frac{1}{4}\left(-8\zeta(2)\ln 2 + 14\zeta(3)\right)\partial^{\mu\nu}\tilde{T}\partial_{\mu\nu}\tilde{T}\partial^\alpha\tilde{T}\partial_\alpha\tilde{T} \\
& \left.+\alpha'^3\frac{7}{2^3 3}\zeta(3)\partial^\mu\tilde{T}\partial_\mu\tilde{T}\partial^\nu\tilde{T}\partial_\nu\tilde{T}\partial^\alpha\tilde{T}\partial_\alpha\tilde{T} + \mathrm{O}(\partial^8)\right). \qquad (33)
\end{aligned}
$$

The coefficients that appear in (33) have been regularized as in the appendix. Since they diverge individually, it is possible to obtain any finite value for the difference between the sums that come from the worldsheet bosons and fermions. The choice of ζ–function regularization, which is equivalent to a choice of field redefinition, fixes these coefficients. Unlike (14) there is term that is quadratic in both derivatives and the field $\tilde{T}$, a standard kinetic term, however unusual normalization is an artifact of this ζ–function regularization procedure. As in the bosonic case it is possible to extend this analysis to higher numbers of derivatives.

We discussed earlier the expression exact in $k_1 \cdot k_2 > 0$ for the two point function. For $k^2 \ll 1$ the relevant expansion can be expressed as a difference of divergent sums. The higher derivative terms that are quadratic in $\tilde{T}$ can be derived or read off from table 2 and the 'kinetic' term with order ∂^{2n} derivatives is given by the regularization of

$$
-\frac{1}{4}\frac{1}{n!}\partial^{\mu_1\ldots\mu_n}\tilde{T}\partial_{\mu_1\ldots\mu_n}\tilde{T}\sum\left[\frac{1}{m_1\ldots m_n} - \frac{n}{2^n}\frac{1}{m_1\ldots m_{n-1}}\frac{1}{r \pm m_1\ldots \pm m_{n-1}}\right]. \qquad (34)
$$

In this, the sum is taken over the positive integers m and half–integers r, and there is an implicit sum over all possible signs in the second denominator following the conventions of the appendix. The first sum can be seen as a

Prefactor	Derivative Form	Associated Sum
$-\frac{1}{4}$	T^2	
$-\frac{1}{4}2\alpha'$	$\partial^\mu T \partial_\mu T$	$\left(\frac{1}{n} - \frac{1}{r}\right)$
$-\frac{1}{8}(2\alpha')^2$	$\partial_{\mu\nu} T \partial^{\mu\nu} T$	$\left(\frac{1}{n}\frac{1}{m} - \frac{1}{n}\frac{1}{r \pm n}\right)$
$\frac{1}{4^2 2^3}(2\alpha')^2$	$\partial_{\mu\nu} T^2 \partial^{\mu\nu} T^2$	$\frac{1}{n^2}$
$-\frac{1}{4^2 2}(2\alpha')^2$	$\partial_\mu T \partial^\mu T \partial_\nu T \partial^\nu T$	$\frac{1}{r^2}$
$-\frac{1}{4 \cdot 3!}(2\alpha')^3$	$\partial_{\mu\nu\alpha} T \partial^{\mu\nu\alpha} T$	$\left(\frac{1}{nn'm} - \frac{3}{2^2}\frac{1}{nn'}\frac{1}{r \pm n \pm n'}\right)$
$\frac{1}{4^2 2!^2 4}(2\alpha')^3$	$\partial_{\mu\nu\alpha} T^2 \partial^{\mu\nu\alpha} T^2$	$\frac{1}{nn'(n+n')}$
$\frac{1}{4^2 2!^2}(2\alpha')^3$	$\partial_{\mu\nu}\left(\partial_\alpha T \partial^\alpha T\right)\partial^{\mu\nu} T^2$	$\frac{1}{n^2 n'}$
$-\frac{1}{4^2 4}(2\alpha')^3$	$\partial_{\mu\nu} T \partial^\mu{}_\alpha T \partial^{\nu\alpha} T^2$	$\frac{1}{n^2}\frac{1}{r \pm n}$
$-\frac{1}{4^2 2!}(2\alpha')^3$	$\partial_\mu T \partial^{\mu\nu\alpha} T \partial_{\nu\alpha} T^2$	$\frac{1}{rn^2}$
$-\frac{1}{4^2 2!}(2\alpha')^3$	$\partial_\mu T \partial^{\mu\nu} T \partial_{\nu\alpha} T \partial^\alpha T$	$\left(\frac{1}{rn}\left(\frac{1}{r-n} + \frac{3}{r+n}\right) + \frac{1}{rr'(r+r')}\right)$
$-\frac{1}{4^2 2!}(2\alpha')^3$	$\partial_{\mu\nu} T \partial^{\mu\nu} T \partial_\alpha T \partial^\alpha T$	$\left(\frac{1}{nr^2} + \frac{1}{n(n+r)^2} - \frac{1}{r^2(r+r')}\right)$
$-\frac{1}{4^3 3! 2}(2\alpha')^3$	$\partial_{\mu\nu} T^2 \partial^{\nu\alpha} T^2 \partial_\alpha{}^\mu T^2$	$\frac{1}{n^3}$
$\frac{1}{4^3 3}(2\alpha')^3$	$\partial^\mu T \partial_\mu T \partial^\nu T \partial_\nu T \partial^\alpha T \partial_\alpha T$	$\frac{1}{r^3}$

Table 2: The terms that appear in the tachyon action for superstring theory in the expansion around the constant tachyon, with prefactors and sums. The sums are over positive integers for $n, m \ldots$ and positive half integers for $r, s \ldots$. The terms $\frac{1}{r \pm n \pm m}$ implicitly sum over all possible combinations of signs.

manifestation of the divergent term in (31), while the second comes from the world–sheet fermions. Naively, both sets are divergent proportional to $\ln \Lambda$, where Λ is the cut–off of the world–sheet theory. Using our conventions, the first two terms in the derivative expansion that are quadratic in $\tilde{T}$ are

$$-\frac{1}{4}\tilde{T}^2 + \alpha' \ln 2 \partial^\mu \tilde{T} \partial_\mu \tilde{T} \tag{35}$$

in agreement with [29].

Having calculated an action in the derivative expansion for the superstring tachyon out to 6 orders in derivatives, we check for consistency with other expressions for tachyon actions that have been derived previously. We start with the observation that in the exactly integrable cases discussed [2, 7, 9], namely the quadratic and linear tachyon profiles for the bosonic and superstring cases respectively. We reproduce the first terms of the exact action, as expected since we reproduce the techniques that were used to

derive these initially.

Similarly, we can compare with the calculation [28] which expands around the constant tachyon background. Taking their expression for the tachyon effective action to order α'^2 in the absence of gauge fields and for a real tachyon, we have

$$\begin{aligned} Z &= T_9 e^{-T^2}\Big[2 + 8\alpha' \ln 2\,(\partial_\mu T)^2 + 4\alpha'^2\gamma_0\,(\partial_\mu\partial_\nu T)^2 \\ &\quad +4\alpha'^2\left(4\,(\ln 2)^2 - \zeta(2)\right)(\partial_\mu T\partial^\mu T)^2 - \alpha'^2\frac{2\pi^2}{3}(\partial_\mu T\partial^\mu T)^2 \\ &\quad +2\zeta(2)\alpha'^2 T\partial^\mu\partial^\nu T\left(\partial_\mu\partial_\nu T^2 + 2\partial_\mu\partial_\nu T\right)\Big]. \end{aligned} \tag{36}$$

Here T_9 is the D9 brane tension, T is the renormalized tachyon field of [28], and the parameter γ_0 is defined in the small ϵ limit as the regularized sum

$$\gamma_0 = \sum_{m=1}^{\infty}\sum_{r=\frac{1}{2}}^{\infty}\frac{1}{m}\left(\frac{1}{r+m}+\frac{1}{r-m}\right)e^{-(r+m)\epsilon} - (\ln\epsilon)^2. \tag{37}$$

To show the correspondence with our results (33), we first rewrite

$$(\ln\epsilon)^2 = \sum_{m,n=1}^{\infty}\frac{1}{mn}e^{-\epsilon(m+n)} = \sum_{m,n=1}^{\infty}\frac{2}{m(m+n)}e^{-\epsilon(m+n)} \tag{38}$$

and then see that in the $\epsilon \to 0$ limit (37) is our sum (A.10) and so

$$\gamma_0 = -4(\ln 2)^2 + 2\zeta(2) \tag{39}$$

as detailed in the appendix. Using this regularization, the terms from (36) match those that we obtain. In addition, the off–shell field redefinition (10) is seen in the renormalized field T_R in [28]. The field redefinition is ubiquitous, and so appears in any kind of string effective action, not only for tachyons, but also for massless and massive modes. This is a rescaling of the boundary fields on–shell. Whether the rescaling enhances or suppresses the field is determined by the field's worldsheet dimension (relevance, irrelevance or marginality).

4 Discussion and Conclusions

In this note, we have examined some of the issues associated with the derivative expansion of the (super)string tachyon action. In particular, we wished

to focus on the compatibility of the derivative expansion in different momentum regimes. To this end, we have given an expression for the partition function of the bosonic string coupled to a single tachyon momentum mode (11). This interpolates between the well–known results for the constant and rolling tachyons [2, 21]. This also reproduces the spatial derivative expansion around the rolling tachyon derived by [22, 17]. Our expansion around the constant tachyon (14) reproduces the first terms of an exact result for the three tachyon coupling calculated by [30]. We have identified a field redefinition (10) which reproduces that of [22, 28] for the world–sheet cut–off dependent coefficients in the tachyon action. This has been noticed by other authors [30, 31], however, we believe it worth remarking on because of the prevalence in the literature of this kind of term. For the case of the superstring tachyon, we have extended by two orders in derivatives the calculations of [28]. Our calculation includes a hint about the purpose of the contact terms in the worldsheet theory. These contact terms are necessary for unbroken world–sheet supersymmetry and contribute to some of the originally calculated amplitudes [6], but are absent in the context of calculations around the rolling tachyon. The reason for this seems to be that the tachyon correlation functions are similar to β functions, ratios of Γ functions. These have a particular domain of convergence for the integrals involved, and can be analytically continued to other regions. The contact terms seem to act to regularize the integrals that arise, and here we have presented the circumstantial evidence that the terms in the derivative expansion of the tachyon effective action, while some are naively divergent, admit a ζ–function regularization which matches with exact results.

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A Sums and Regularization Methods

Here we tabulate some useful sums that appear in the world–sheet approach to this problem. Throughout this we will use the convention that $m, n \ldots$ are positive integers and $r, s \ldots$ are positive half–integers. We use a ζ-function

regularization, with the definition of Riemann's ζ-function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{A.1}$$

This has a pole at $z = 1$, and an analytic continuation throughout the complex z plane. This can also be related to a sum over the positive half-integers, as

$$\begin{aligned} \sum_{r=1/2}^{\infty} \frac{1}{r^z} &= \sum_{m=1}^{\infty} \frac{2^z}{(2m-1)^z} + \sum_{m=1}^{\infty} \frac{2^z}{(2m)^z} - \zeta(z) \\ &= (2^z - 1)\,\zeta(z). \end{aligned} \tag{A.2}$$

A related function which will be useful for our purposes is the ψ function, defined as

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x), \tag{A.3}$$

with the series representation

$$\psi(x) = -\gamma - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}, \tag{A.4}$$

where γ is Euler's constant, $\gamma \approx 0.5772$. The derivatives of $\psi(x)$ satisfy

$$x\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=1}^{\infty} \frac{1}{(x+k)^{n+1}}. \tag{A.5}$$

We also note that with the help of the integral $\int_0^1 x^m (\ln x)^n dx = \frac{n!}{(m+1)^{n+1}}$ the sums of the form

$$\sum_{m_1 \ldots m_n = 1}^{\infty} \frac{1}{m_1 \ldots m_n (m_1 + \ldots m_n)} = n! \zeta(n+1) \tag{A.6}$$

can be evaluated.

In table 1 we list two sums that are not covered by the above considerations

$$\begin{aligned} \sum \frac{4}{mnn'(m+n+n')} + \frac{3}{mm'nn'} \delta_{m+m',n+n'} &= 8 \oint \frac{d\phi}{2\pi} |\ln(2 \sin \frac{\phi}{2})|^4 \\ &= 6\left(\zeta(2)^2 + 7\zeta(4)\right), \end{aligned} \tag{A.7}$$

$$\begin{aligned} \sum \frac{2}{n^2 m(m+n)} + \frac{1}{mn(m+n)^2} &= \zeta(2)^2 + 2 \sum \frac{1}{m(m+n)^3} \\ &= \zeta(2)^2 + \frac{1}{2}\zeta(4). \end{aligned} \tag{A.8}$$

For the superstring, we give explicitly our regularizations of the sums that appear in the expansion of the tachyon bilinear term in powers of momentum.

$$\begin{aligned}\sum \frac{1}{n} - \sum \frac{1}{r} &= \sum \left(\frac{1}{m} - \frac{1}{m-\frac{1}{2}} \right) \\ &= \lim_{x\to 1} 2 \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \\ &= -2\ln 2 \end{aligned} \tag{A.9}$$

This can alternatively be obtained using representations of the infinite sums as logarithmic derivatives of the Γ–function. In the case of 4 derivatives we find

$$\begin{aligned}\sum \left(\frac{1}{mn} - \frac{1}{m(r\pm m)} \right) &= \sum \left(\frac{2}{m(n+m)} - \frac{1}{m(r+m)} - \frac{1}{m(r-m)} \right) \\ &= -\sum \frac{2}{m} \left(\psi(m+1) - \psi(m+\frac{1}{2}) \right) \\ &= -4 \lim_{x\to 1} \sum \frac{x^{2m}}{m} \left(\ln 2 + \sum_{k=1}^{m} \frac{1}{2k} - \frac{1}{2k-1} \right) \\ &= -4 \lim_{x\to 1} \sum \frac{x^{2m}}{m} \left(\ln 2 + \frac{1}{2m} + \sum_{k=1}^{2m-1} \frac{(-1)^k}{k} \right) \\ &= -2\zeta(2) + 4(\ln 2)^2 \end{aligned} \tag{A.10}$$

We have abbreviated $\frac{1}{m(r\pm m)} \equiv \frac{1}{m(r+m)} + \frac{1}{m(r-m)}$ as mentioned in table 2. In (A.10) we have used the symmetry of the ψ, namely, $\psi(n+\frac{1}{2}) = \psi(\frac{1}{2}-n)$ for $n \in \mathbb{Z}$, and the identity $\ln(1-x)\ln(1+x) = \sum_k \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^n}{n}$. In the 6 derivative case we find

$$\begin{aligned}\sum \frac{1}{nn'm} - \frac{3}{2^2}\frac{1}{nm}\frac{1}{r\pm n\pm m} &= 3\sum \frac{1}{nm(n'+m+n)} - \frac{1}{4}\frac{1}{nm(r\pm n\pm m)} \\ &= 3\sum \frac{1}{mn}\psi(1+m+n) - \frac{1}{4}\psi(\frac{1}{2}\pm n\pm m) \\ &= 3\sum_{m,n=1}^{\infty} \frac{1}{mn} \left(2\ln 2 + \sum_{k=1}^{m+n} \frac{1}{k} \right. \\ &\qquad \left. - \sum_{k=1}^{m+n} \frac{1}{2k-1} - \sum_{k=1}^{|m-n|} \frac{1}{2k-1} \right) . \end{aligned} \tag{A.11}$$

In this sum, the third and fourth term can be related by the observations that

$$\begin{aligned}\sum_{n,p=1}^{\infty}\frac{1}{np}\sum_{m=1}^{n+p}\frac{1}{2m-1} &= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\frac{1}{n(l-n)}\sum_{m=1}^{l}\frac{1}{2m-1} \\ &= 2\sum_{l=1}^{\infty}\left(\sum_{n=1}^{l}\frac{1}{ln}-\frac{1}{l^2}\right)\sum_{m=1}^{l}\frac{1}{2m-1}, \qquad \text{(A.12)}\end{aligned}$$

and

$$\begin{aligned}\sum_{n,p=1}^{\infty}\frac{1}{np}\sum_{m=1}^{|n-p|}\frac{1}{2m-1} &= 2\sum_{\substack{n,p=1\\ n>p}}^{\infty}\frac{1}{np}\sum_{m=1}^{n-p}\frac{1}{2m-1} \\ &= 2\sum_{n=2}^{\infty}\sum_{l=1}^{n-1}\frac{1}{n(n-l)}\sum_{m=1}^{l}\frac{1}{2m-1} \\ &= 2\sum_{l=1}^{\infty}\frac{1}{l}\sum_{n=1}^{l}\frac{1}{n}\sum_{m=1}^{l}\frac{1}{2m-1}, \qquad \text{(A.13)}\end{aligned}$$

from which we see that the two sums are related as

$$\sum_{n,p=1}^{\infty}\frac{1}{np}\sum_{m=1}^{|n-p|}\frac{1}{2m-1}=\sum_{n,p=1}^{\infty}\frac{1}{np}\sum_{m=1}^{n+p}\frac{1}{2m-1}+2\sum_{l=1}^{\infty}\frac{1}{l^2}\sum_{m=1}^{l}\frac{1}{2m-1}. \qquad \text{(A.14)}$$

This allows us to regularize

$$\begin{aligned}\sum\frac{1}{nn'm}-\frac{3}{2^2}\frac{1}{nm}\frac{1}{r\pm n\pm m} &= 3\sum\frac{2}{mn}\left(\sum_{k=1}^{2(n+m)}\frac{(-1)^k}{k}+2\ln 2\right) \\ &\quad -6\sum\frac{1}{m^2}\sum_{k=1}^{m}\frac{1}{2k-1} \\ &= 6\int_0^1 dx\left(\frac{\ln(1-x^2)^2}{1+x}-\frac{\ln(1-x^2)^2}{2x}\right. \\ &\quad \left.+2\frac{\ln(1+x)\ln(1-x)}{x}\right) \\ &= 8(\ln 2)^3+\frac{3}{2}\zeta(3)-12\zeta(2)\ln 2. \qquad \text{(A.15)}\end{aligned}$$

Using similar manipulations we can obtain:

$$
\begin{aligned}
\sum \frac{1}{nn'(n+n')} &= 2\zeta(3) && \text{(A.16)}\\
\sum \frac{1}{n(n+n')^2} &= \zeta(3) && \text{(A.17)}\\
\sum \frac{1}{rr'(r+r')} &= 7\zeta(3) && \text{(A.18)}\\
\sum \frac{1}{n(n+r)^2} &= -6\zeta(2)\ln 2 + 7\zeta(3) && \text{(A.19)}\\
\sum \frac{1}{r(n+r)^2} &= 6\zeta(2)\ln 2 - \frac{7}{2}\zeta(3) && \text{(A.20)}\\
\sum \frac{1}{m^2}\left(\frac{1}{n} - \frac{1}{2}\frac{1}{r \pm m}\right) &= \frac{7}{2}\zeta(3) - 2\zeta(2)\ln 2 && \text{(A.21)}\\
\sum \frac{1}{rn(r+n)} &= \frac{7}{2}\zeta(3) && \text{(A.22)}\\
\sum \frac{1}{rn}\left(\frac{1}{r+n} - \frac{1}{r-n}\right) &= 7\zeta(3) && \text{(A.23)}\\
\sum \frac{1}{r^2}\left(\frac{1}{n} - \frac{1}{r+r'}\right) &= -6\zeta(2)\ln 2 + 7\zeta(3). && \text{(A.24)}
\end{aligned}
$$

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